Online Appendix of A Continuous-Time Macro-Finance Model with Knightian Uncertainty *

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A Proof of Proposition 1

Guess that the value function $V_t = \theta_t n_t$,we have differential value functions $V'_\theta = n_t$, $V'_\theta = \theta_t$, and $V''_{n\theta} = 1$.By plugging them in Eq. (21), the HJBI equation, we get

$$\begin{aligned} \rho dt = \sup_{d\zeta_t \geq 0, x_t \geq 0} \inf_{h_t \in [-\Delta, \Delta]} & \frac{1-\theta_t}{\theta_t} d\zeta_t + (\mu_t^\theta + h_t \sigma_t^\theta) dt + r(1-x_t) dt \\ & + x_t \left\{ \frac{\alpha - \iota_t}{q_t} + [\Phi(\iota_t) - \delta + h_t \sigma + \mu_t^q + \sigma \sigma_t^q] + \sigma_t^\theta (\sigma + \sigma_t^q) \right\} dt, \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} \rho dt = \sup_{d\zeta_t \geq 0, x_t \geq 0} \inf_{h_t \in [-\Delta, \Delta]} & \frac{1-\theta_t}{\theta_t} d\zeta_t + \mu_t^\theta dt + r(1-x_t) dt + h_t [\sigma_t^\theta + x_t(\sigma + \sigma_t^q)] dt \\ & + x_t \left\{ \frac{\alpha - \iota_t}{q_t} + [\Phi(\iota_t) - \delta + h_t \sigma + \mu_t^q + \sigma \sigma_t^q] + \sigma_t^\theta (\sigma + \sigma_t^q) \right\} dt. \end{aligned} \tag{A.2}$$

*For detailed and helpful comments, we are thankful to anonymous referees, Tak-Yuen Wong, Qihong Chen, and numerous seminar participants. For academic and financial supports, Mao thanks China Postdoctoral Science Foundation (No.2019M661386) and Shanghai Super Postdoctoral Science Foundation (No. 2019159), and Yan thanks Scientific Research Staring Foundation of Sichuan University.

Volatility Terms σ , σ_t^q, and σ_t^θ are greater than 0. Meanwhile, $x_t \geq 0$. As a result, $\sigma_t^\theta + x_t(\sigma + \sigma_t^q) \geq 0$, which means that the value of h_t is to be $-\Delta$. Thus, we have

$$\begin{aligned}\rho dt = \sup_{d\zeta_t \geq 0, x_t \geq 0} & \frac{1-\theta_t}{\theta_t} d\zeta_t + (\mu_t^\theta - \Delta\sigma_t^\theta)dt + r(1-x_t)dt \\ & + x_t \left\{ \frac{\alpha - \iota_t}{q_t} + [\Phi(\iota_t) - \delta + h_t\sigma + \mu_t^q + \sigma\sigma_t^q] + \sigma_t^\theta(\sigma + \sigma_t^q) \right\} dt.\end{aligned} \tag{A.3}$$

θ_t should meet two conditions. The first condition is that $\theta_t \geq 1$. Especially, the marginal utilities of wealth equals that of consumption when $\theta_t = 1$. In that case, $d\zeta_t > 0$. The second condition is that $\mathbb{E}[e^{-\rho t}\theta_t n_t] \to 0$, namely the Non-Ponzi condition. Holding $d\zeta_t = 0$ and $x_t = 0$,the representative expert's marginal utility of wealth θ_t's drift term μ_t^θ satisfies

$$\mu_t^\theta = \rho + \Delta\sigma_t^\theta - r. \tag{A.4}$$

By taking derivatives of both sides of Eq. (A.4) with respect to risk asset ratio x_t , we know that θ_t's volatility term σ_t^θ satisfies

$$\sigma_t^\theta = -\frac{\frac{\alpha - \iota_t}{q_t} + \Phi(\iota_t) - \delta - \Delta\sigma + \mu_t^q - \Delta\sigma_t^q + \sigma\sigma_t^q - r}{\sigma + \sigma_t^q}. \tag{A.5}$$

Therefore, if experts hold capital ($x_t > 0$), σ_t^θ equals the Sharp ratio of holding capital:

$$\frac{\alpha - \iota_t}{q_t} + \Phi(\iota_t) - \delta - \Delta\sigma + \mu_t^q - \Delta\sigma_t^q + \sigma\sigma_t^q - r = -\sigma_t^\theta(\sigma + \sigma_t^q). \tag{A.6}$$

If experts hold no capital ($x_t = 0$), σ_t^θ is smaller than the Sharp ratio:

$$\frac{\alpha - \iota_t}{q_t} + \Phi(\iota_t) - \delta - \Delta\sigma + \mu_t^q - \Delta\sigma_t^q + \sigma\sigma_t^q - r < -\sigma_t^\theta(\sigma + \sigma_t^q). \tag{A.7}$$

In summary, the marginal utility of wealth θ_t should meet the following condition:

$$\frac{\alpha - \iota_t}{q_t} + \Phi(\iota_t) - \delta - \Delta\sigma + \mu_t^q - \Delta\sigma_t^q + \sigma\sigma_t^q - r \leq -\sigma_t^\theta(\sigma + \sigma_t^q). \tag{A.8}$$

In Eq. (A.8), the equality holds when $x_t > 0$.

The following demonstrates the rationality of guessing that $V_t = \theta_t n_t$.

Under the measure $\mathbb{P}$, an expert would choose its optimal consumption ratio $d\zeta_t$ and risk asset share x_t in the worst case. It is assumed that the representative expert's value function:

$$\theta_t n_t = \sup_{d\zeta_t \geq 0, x_t \geq 0} \inf_{h_t \in [-\Delta, \Delta]} \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} dc_s \right], \tag{A.9}$$

subject to the Eq. (17) and (19). The process $e^{-\rho t}\theta_t n_t + \int_0^t e^{-\rho s} dc_s$ is a martingale under the optimal choice and the worst case. Using Itô's lemma, we conclude that the expert problem satisfies the following Hamilton–Jacobi–Bellman–Isaac equations:

$$
\begin{aligned}
\rho\theta_t n_t dt = & \sup_{d\zeta_t \geq 0, x_t \geq 0} \inf_{h_t \in [-\Delta,\Delta]} \mathbb{E}[dc_t] + \mathbb{E}[d(\theta_t n_t)] \\
& = \sup_{d\zeta_t \geq 0, x_t \geq 0} \inf_{h_t \in [-\Delta,\Delta]} n_t d\zeta_t + (\mu_t^\theta - h_t\sigma_t^\theta)\theta_t dt \\
& + x_t \left(\frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta - h_t\sigma + \mu_t^q - h_t\sigma_t^q + \sigma\sigma_t^q \right) n_t\theta_t dt \\
& + (r(1 - x_t) - d\zeta_t/dt) n_t\theta_t dt + n_t\theta_t x_t \sigma_t^\theta(\sigma + \sigma_t^q) dt \,.
\end{aligned}
\tag{A.10}
$$

Next, prove that the following equation holds

$$
\theta_t n_t = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} dc_s \right]. \tag{A.11}
$$

Consider the process:

$$
M_t = e^{-\rho t}\theta_t n_t + \int_0^t e^{-\rho s} dc_s. \tag{A.12}
$$

By differentiating M_t with respect to t, and applying Itô's lemma, we have

$$
\begin{aligned}
dM_t &= d(e^{-\rho t}\theta_t n_t) + d\left(\int_0^t e^{-\rho s} dc_s \right) \\
&= -\rho e^{-\rho t}\theta_t n_t dt + e^{-\rho t} d(\theta_t n_t) + e^{-\rho t} dc_t \\
&= e^{-\rho t}(-\rho\theta_t n_t dt + d(\theta_t n_t) + dc_t).
\end{aligned}
\tag{A.13}
$$

If $\rho\theta_t n_t = dc_t + \mathbb{E}[d(\theta_t n_t)]$ holds, then $\mathbb{E}[dM_t] = 0$, so M_t is a martingale under the optimal strategy (ζ_t, x_t) and the worst case h_t. Therefore,

$$
\theta_0 n_0 = M_0 = \mathbb{E}[M_t] = \mathbb{E}[e^{-\rho t}\theta_t n_t] + \mathbb{E}\left[\int_0^t e^{-\rho s} dc_s \right]. \tag{A.14}
$$

Taking the limit $t \to \infty$ and using the transversality condition $\mathbb{E}[e^{-\rho t}\theta_t n_t] \to 0$, we have

$$
\theta_0 n_0 = \mathbb{E}\left[\int_0^\infty e^{-\rho s} dc_s \right]. \tag{A.15}
$$

Similar to the calculation of 0, we can ascertain that this equation is valid at any time t.

In contrast, according to equation (A.11) for the optimal strategy and the worst case, we have

$$
e^{-\rho t}\theta_t n_t = \mathbb{E}_t \left[\int_t^\infty e^{-\rho s} dc_s \right]. \tag{A.16}
$$

Add $\int_0^t e^{-\rho s} dc_s$ to both sides of this equation, then,

$$
e^{-\rho t}\theta_t n_t + \int_0^t e^{-\rho s} dc_s = \mathbb{E}_t \left[\int_t^\infty e^{-\rho s} dc_s + \int_0^t e^{-\rho s} dc_s \right]. \tag{A.17}
$$

So, we have

$$M_t = e^{-\rho t}\theta_t n_t + \int_0^t e^{-\rho s} dc_s = \mathbb{E}_t\left[\int_0^\infty e^{-\rho s} dc_s\right], \tag{A.18}$$

then M_t is a martingale. Therefore, the drift of M_t must be zero, and so $\rho\theta_t n_t = dc_t + \mathbb{E}[d(\theta_t n_t)]$ holds under the optimal strategy and the worst case.

Next, we will demonstrate that the strategy $\{x, d\zeta_t\}$ is optimal if and only if the Bellman equation (A.10) holds. Under any alternative strategy $\{x, d\zeta_t\}$, define the following process:

$$M_t = e^{-\rho t}\theta_t n_t + \int_0^t e^{-\rho s} dc_s. \tag{A.19}$$

Fix the process h^* for the worst case. By Itô's lemma under the probability measure $\mathbb{P}$,

$$\begin{aligned} e^{\rho t} dM_t &= G_t^{(x,d\zeta,h)} dt - \rho\theta_t n_t dt + (1-\theta_t) n_t d\zeta_t \\ &\leq G_t^{(x^*,d\zeta^*,h^*)} dt - \rho\theta_t n_t dt + (1-\theta_t) n_t d\zeta_t \leq 0, \end{aligned} \tag{A.20}$$

where

$$\begin{aligned} G_t^{(x,d\zeta,h)} &= (\mu_t^\theta - h_t\sigma_t^\theta)\theta_t + r(1-x_t)n_t\theta_t + n_t\theta_t x_t\sigma_t^\theta(\sigma + \sigma_t^q) \\ &\quad + x_t\left(\frac{a-\iota_t}{q_t} + \Phi(\iota_t) - \delta - h_t\sigma + \mu_t^q - h_t\sigma_t^q + \sigma\sigma_t^q\right) n_t\theta, \end{aligned} \tag{A.21}$$

the HJBI equation (A.10) holds, then M_t is a supermartingale under an arbitrary alternative strategy, this implies that

$$M_0 \;\; \geq \;\; \mathbb{E}\left[M_{t\wedge\tau}\right]. \tag{A.22}$$

For any finite time $t \geq 0$, taking limit as $t \to \infty$, we have

$$M_0 \;\; \geq \;\; \mathbb{E}\left[M_\tau\right] \geq \inf_{h\in[-\Delta,\Delta]} \mathbb{E}\left[M_\tau\right]. \tag{A.23}$$

Taking supremum for $(x, d\zeta_t)$ and using (A.19), we obtain

$$\theta_0 n_0 = M_0 \geq \sup_{d\zeta\geq 0, x\geq 0} \mathbb{E}\left[M_\tau\right] \geq \sup_{d\zeta\geq 0, x\geq 0} \inf_{h\in[-\Delta,\Delta]} \mathbb{E}\left[M_\tau\right]. \tag{A.24}$$

Fixing $(x^*, d\zeta_t^*)$ and consider any process (h_t) , we use Itô's lemma to derive

$$\begin{aligned} e^{\rho t} dM_t &= G_t^{(x,d\zeta,h)} dt - \rho\theta_t n_t dt + (1-\theta_t) n_t d\zeta_t \\ &\geq G_t^{(x^*,d\zeta^*,h^*)} dt - \rho\theta_t n_t dt + (1-\theta_t) n_t d\zeta_t \geq 0. \end{aligned} \tag{A.25}$$

Note that $G_t^{(x^*,d\zeta^*,h^*)} - \rho\theta_t n_t = 0$. Thus M_t is a submartingale. This implies that

$$M_0 \;\; \leq \;\; \mathbb{E}\left[M_{t\wedge\tau}\right]. \tag{A.26}$$

For any finite time t, taking limit as $t \to \infty$, we have

$$n_0\theta_0 \le M_0 \le \mathbb{E}\left[M_\tau\right]. \tag{A.27}$$

Taking infimum for h and using (A.19), we obtain

$$\theta_0 n_0 = M_0 \le \inf_{h\in[-\Delta,\Delta]} \mathbb{E}\left[M_\tau\right] \le \sup_{d\zeta\ge 0, x\ge 0} \inf_{h\in[-\Delta,\Delta]} \mathbb{E}\left[M_\tau\right]. \tag{A.28}$$

Thus, we deduce that

$$\theta_0 n_0 = M_0 = \sup_{d\zeta\ge 0, x\ge 0} \inf_{h\in[-\Delta,\Delta]} \mathbb{E}\left[M_\tau\right]. \tag{A.29}$$

B Proof of Proposition 2

Using Itô's lemma, $q_t = q(\eta_t)$ can be transformed into

$$dq_t = \left[q'(\eta_t)\mu_t^\eta \eta_t + \frac{1}{2}q''(\eta_t)(\sigma_t^\eta)^2(\eta_t)^2\right]dt + q'(\eta_t)\sigma_t^\eta \eta_t dW_t. \tag{B.1}$$

Consequently,

$$\mu_t^q = \frac{q'(\eta_t)\mu_t^\eta \eta_t + \frac{1}{2}q''(\eta_t)(\sigma_t^\eta)^2(\eta_t)^2}{q_t}, \tag{B.2}$$

$$\sigma_t^q = \frac{q'(\eta_t)\mu_t^\eta \eta_t}{q_t} = \frac{q'(\eta_t)}{q_t}(\psi_t - \eta_t)(\sigma + \sigma_t^q) = \frac{(\psi_t - \eta_t)q'(\eta_t)/q(\eta_t)}{1 - (\psi_t - \eta_t)q'(\eta_t)/q(\eta_t)}\sigma. \tag{B.3}$$

Combine Eq. (31) with Eq. (B.3), we have

$$\sigma_t^\eta = \frac{\psi_t/\eta_t - 1}{1 - (\psi_t - \eta_t)q'(\eta_t)/q(\eta_t)}\sigma, \tag{B.4}$$

$$\sigma_t^q = \frac{q'(\eta_t)}{q(\eta_t)}\eta_t \sigma_t^\eta. \tag{B.5}$$

Similarly, we have

$$\mu_t^\theta = \frac{\theta'(\eta_t)\mu_t^\eta \eta_t + \frac{1}{2}\theta''(\eta_t)(\sigma_t^\eta)^2(\eta_t)^2}{\theta_t}, \tag{B.6}$$

$$\sigma_t^\theta = \frac{\theta'(\eta_t)}{\theta(\eta_t)}\eta_t \sigma_t^\eta. \tag{B.7}$$

If the wealth share held by experts falls to 0 ($\eta = 0$), experts would have to liquidate their capital. In that case, the capital price would be the liquidation price $\underline{q}$, i.e.,

$$q(0) = \underline{q}. \tag{B.8}$$

At this point, the utility of each expert is infinite, i.e.,

$$\lim_{\eta_t \to 0} \theta(\eta_t) = +\infty. \tag{B.9}$$

Define η^* as the critical wealth share at which experts choose to consume, i.e.,

$$\theta(\eta^*) = 1. \tag{B.10}$$

Meanwhile, $q(\eta_t)$ and $\theta(\eta_t)$ satisfy the smooth contact condition when $\eta = \eta^*$, i.e.,

$$q'(\eta^*) = 0, \tag{B.11}$$

$$\theta'(\eta^*) = 0. \tag{B.12}$$

C Proof of Proposition 3

$T_{\eta_0}(\eta)$ denotes the expected time it takes to reach a point η_0 starting from $\eta \geq \eta_0$. To reach η_0 from η^*, one has to reach $\eta \in (\eta_0, \eta^*)$ first and then reach η_0 from η.Therefore,

$$\tau(\eta) = \tau(\eta_0) - T_{\eta_0}(\eta). \tag{C.1}$$

Since $t + T_{\eta_0}(\eta)$ is a martingale, $T_{\eta_0}(\eta)$ shall satisfy the ordinary differential equation

$$1 + \mu_t^\eta \eta T'_{\eta_0}(\eta) + \frac{1}{2}(\sigma_t^\eta \eta)^2 T''_{\eta_0}(\eta) = 0. \tag{C.2}$$

As $\tau'(\eta) = -T'_{\eta_0}(\eta)$ and $\tau''(\eta) = -T''_{\eta_0}(\eta)$, $\tau(\eta)$ meets

$$1 - \mu_t^\eta \eta \tau'(\eta) + \frac{1}{2}(\sigma_t^\eta \eta)^2 \tau''(\eta) = 0. \tag{C.3}$$

with boundary value conditions $\tau(\eta^*) = 0$ and $\tau'(\eta^*) = 0$.

D Algorithms for Numerical Solution

Assuming that we know η, $q(\eta)$, $q'(\eta)$, $\theta(\eta)$, $\theta'(\eta)$ and have a guess of $\psi(\eta)$, the goal of the algebra is to compute $q''(\eta)$ and $\theta''(\eta)$, and check whether the guess of $\psi(\eta)$ was correct or not. The algorithm is summarized as follows.

Algorithm 1. Start with η, $q(\eta)$, $q'(\eta)$, $\theta(\eta)$, $\theta'(\eta)$. Search for an appropriate value of $\psi \in \{\eta, \min[1, q(\eta)/q'(\eta) + \eta]\}$ via the following procedure.

(i) Set $\psi_L = \eta$ and $\psi_H = \min[1, q(\eta)/q'(\eta) + \eta]$. Guess that $\psi = (\psi_L + \psi_H)/2$.

(ii) Calculate σ_t^q, σ_t^η, σ_t^θ and μ_t^q from Eq. (42) to Eq. (47).

(iii) If Eq. (24) is satisfied, adjust the guess of ψ by setting $\psi_H = \psi$. Otherwise, adjust the guess of ψ by setting $\psi_L = \psi$.

(iv) Repeat the step (ii) and (iii) for 30 times.

After step (iv), we can find an appropriate value of ψ. Based on this value, we execute Eq. Eq. (42) to Eq. (47) to get $q''(\eta)$ and $\theta''(\eta)$.

In Algorithm 1, the numerical computation of the functions $q(\eta)$, $\theta(\eta)$ and $\psi(\eta)$ poses several challenges. The first one relates to the singularity at $\eta = 0$. Secondly, we need to determine the endogenous endpoint η^* and match the boundary conditions at both 0 and η^*. Fortunately, it is helpful to observe that, given function $\theta(\eta)$ solves the equations of Proposition 2, function $\xi\theta(\eta)$ solves Proposition 2 for any constant $\xi > 0$. Therefore, it is feasible to adjust the level of $\theta(\eta)$ ex post to match the boundary condition.

Algorithm 2 performs an appropriate search and effectively addresses the singularity issue by solving the system of equations with the boundary condition $\theta(0) = M$, for a large constant M, instead of Eq. (38).

Algorithm 2. Set

$$q(0) = \underline{q} = \max_{\underline{\iota_t}} = \frac{\underline{\alpha} - \underline{\iota_t}}{r - [\Phi(\underline{\iota_t}) - \underline{\delta} - \Delta\sigma]}, \theta(0) = 1 \text{ and } \theta'(0) = -10^{10}. \tag{D.1}$$

Perform the following procedure to find an appropriate boundary condition $q'(0)$.

(i) Set $q_L = 0$ and $q_H = 10^{15}$. Guess that $q'(0) = (q_L + q_H)/2$.

(ii) Use ode45[1] to solve for $q(\eta)$ and $\theta(\eta)$ on the interval $[0, \eta^*]$ until one of the following events happens:(a) $q(\eta)$ reaches the upper bound $\max_{\iota_t} \frac{(\alpha - \iota_t)}{r - [\Phi(\iota_t) - \delta - \Delta\sigma]}$; (b) $\theta'(\eta)$ reaches 0; (c) $q'(\eta)$ reaches 0.

(iii) If step (ii) is terminated for reason (c), increase the guess of $q'(0)$ by setting $q_L = q'(0)$. Otherwise, decrease the guess of $q'(0)$ by setting $q_H = q'(0)$.

(iv) Repeat the step (ii) and (iii) until convergence.

If the initial value of q_H is sufficiently large, $\theta'(\eta)$ and $q'(\eta)$ should eventually reach 0 at the same point, which we denote by η^*. Divide the entire function $\theta(\eta)$ by $\theta(\eta^*)$, then, the boundary condition $\theta(\eta^*) = 1$ is met.

[1] An ODE solver in MATLAB.